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REGULAR NEAR POLYGONS DO CONTAIN HEXES

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Regular near polygons do contain hexes *)

by

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ABSTRACT

It is shown that in a regular near n-gon (with $s > 1$ and $t_2 > 0$) any two points at distance three determine a unique geodetically closed sub near hexagon.

KEY WORDS & PHRASES: *near hexagon, near n-gon*

*) This report will be submitted for publication elsewhere.

INTRODUCTION

A *near polygon* is a partial linear space (X,L) such that for any point $p \in X$ and line $\ell \in L$ there is a unique point on ℓ nearest p. A *regular* near polygon with parameters $(s,t_2,t_3,\ldots,t_d)$ is a near polygon of diameter d such that all lines have s+1 points, each point is on t+1 lines, and any point at distance i from a given point x_0 is adjacent to t_i+1 points at distance i-1 from x_0. (Here distances and adjacency are interpreted in the point graph: two points are adjacent iff they are collinear.) Note that $t_0 = -1$, $t_1 = 0$, $t_d = t$.

SHULT & YANUSHKA [5] studied these objects and showed that if lines have more than two points then any two points at distance two with at least two common neighbours determine a geodetically closed sub generalized quadrangle containing these two points. (A subset Y of X is called *geodetically closed* if it contains all vertices occurring in the shortest paths between any two of its elements.)

One would expect that any two points at distance i determine a geodetically closed sub near 2i-gon under some mild conditions. CAMERON [2] proved that this true under the additional assumption that for any point x and any quad Q there is a unique point in Q closest to x (in the regular case this is equivalent to $t_{i+1} = t_2(t_i+1)$ for $0 \le i \le d-1$) - in fact he showed that a near polygon satisfying this assumption is a dual polar space. SHAD & SHULT [4] proved that the weaker assumption that for any point x and quad Q such that $d(x,Q) = 2$ there is a unique point in Q closest to x (i.e. $t_3 = t_2(t_2+1)$ in the regular case) already implies that two points at distance three determine a geodetically closed sub near hexagon, called a *hex*. Here we prove that one has hexes in arbitrary regular near polygons. For more details on regular near polygons see BROUWER & WILBRINK [1].

NOTATION. $\Gamma_i(x)$ denotes the set of points at distance i from the point x. $x \sim y$ means that x and y are adjacent (i.e., collinear).

0. CONNECTIVITY OF QUADS

LEMMA. *Let Q be a generalized quadrangle with at least three points on each*

line and at least two lines through each point.
(i) If $x \in Q$ *then* $Q \cap \Gamma_2(x)$ *is connected.*
(ii) If $O \subset Q$, O *an ovoid in* Q *then* $Q \backslash O$ *is connected.*

PROOF. (i) Let $y,z \in Q_0 := Q \cap \Gamma_2(x)$. Suppose that y and z are not joined by a path of length at most two in Q_0. Now y is on two lines, so we find two common neighbours u,v of y and z which must be neighbours of x as well. Choose a third point w on the line uy. Then w has a neighbour q on the line zv, and $q \neq v$ so that ywqz is a path in Q_0 joining y and z.
(ii) Let $y,z \in Q_1 := Q \backslash O$. Suppose that y and z are not joined by a path of length at most two in Q_1. Again we find points $u,v \in \Gamma_1(y) \cap \Gamma_1(z) \cap O$ and w on yu and q on zv so that the path ywqz in Q_1 joins y to z. □

1. COMPONENTS OF $\Gamma_3(\infty)$

Let (X,L) be a regular near n-gon with $s > 1$ and $t_2 > 0$.

DEFINITION. Let $d(x,y) = i$. Then $S(x,y) := \{\ell \in L \mid x \in \ell \text{ and } \ell \cap \Gamma_{i-1}(y) \neq \emptyset\}$.

LEMMA. *Let* C *be a component of* $\Gamma_i(x)$. *Then for* $y,y' \in C$ *we have* $S(x,y) =$
$= S(x,y')$.

PROOF. We may assume $y \sim y'$. Let $\ell \in S(x,y)$ and $z \in \ell$, $d(z,y) = i-1$. Now $d(z,y') \leq i$ and $d(x,y') = i$ so there is a point at distance i-1 from y' on the line $xz = \ell$. □

LEMMA. *Let* $\ell,m \in S(x,y)$ *and let* n *be a line on* x *in the quad* $Q(\ell,m)$. *Then* $n \in S(x,y)$.

PROOF. y is either of ovoid type at distance i-1 from $Q = Q(\ell,m)$, or of classical type at distance i-2 from Q. In both cases every line of Q carries a point of $\Gamma_{i-1}(y)$. □

This lemma shows that we may regard S(x,y) as a linear space with the lines through x as points and the quads on x as lines. Since we are supposing regularity this is a $2-(t_i+1,t_2+1,1)$ design. All these designs are subdesigns of the $2-(t+1,t_2+1,1)$ design ('the local design at x') obtained by taking

$d(x,y) = d$.

LEMMA. *Let* $\ell_1,\ldots,\ell_i$ *be* i *lines on* x. *Then there is a* $y \in \Gamma_i(x)$ *such that* $\{\ell_1,\ldots,\ell_i\} \subset S(x,y)$.

PROOF. Suppose $\{\ell_1,\ldots,\ell_{i-1}\} \subset S(x,u)$ for some $u \in \Gamma_{i-1}(x)$. We may assume $\ell_i \notin S(x,u)$. Choose a point $z \in \ell_i \backslash \{x\}$.

Now since $t_i > t_{i-1}$ there is a point $y \in \Gamma_1(u) \cap \Gamma_{i-1}(z) \backslash \Gamma_{i-2}(x)$. That y works. □

LEMMA. $\Gamma_3(x)$ *has at least* $\dfrac{(t+1)t(t-t_2)}{(t_3+1)t_3(t_3-t_2)}$ *components.*

PROOF. Any three lines on x not in a quad are in a subspace $S(x,u)$ for some $u \in \Gamma_3(x)$. But a given subspace $S(x,u)$ contains only $(t_3+1)t_3(t_3-t_2)$ such (ordered) triples. Hence there are at least $\dfrac{(t+1)t(t-t_2)}{(t_3+1)t_3(t_3-t_2)}$ subspaces $S(x,u)$, $u \in \Gamma_3(x)$, and since two points from the same component determine the same subspace, there are at least this many components. □

REMARK. Once we have constructed hexes it follows that $\Gamma_4(x)$ has at least $\dfrac{(t+1)t(t-t_2)(t-t_3)}{(t_4+1)t_4(t_4-t_2)(t_4-t_3)}$ components, etc.

LEMMA. $|\Gamma_3(x)| = \dfrac{s^3(t+1)t(t-t_2)}{(t_2+1)(t_3+1)}$.

PROOF. Follows immediately from the definitions. □

THEOREM. *Let* C *be a component of* $\Gamma_3(x)$ *for some* $x \in X$. *Then* $|C| = \dfrac{s^3t_3(t_3-t_2)}{t_2+1}$.

PROOF. In view of the previous two lemmas it suffices to prove

$$|C| \geq \frac{s^3t_3(t_3-t_2)}{t_2+1}.$$

Fix a point $y \in C$. Then

$$(1) \qquad |C \cap \Gamma_0(y)| = 1,$$

and

(2) $$|C \cap \Gamma_1(y)| = (s-1)(t_3+1).$$

If ℓ,m are two lines on y meeting $\Gamma_2(x)$ then $Q = Q(\ell,m)$ is either of classical type w.r.t. x (and $d(x,Q) = 1$) or Q is of ovoid type w.r.t. x (and $d(x,Q) = 2$). In both cases all lines on y in Q meet $\Gamma_2(x)$, so we find $\frac{(t_3+1)t_3}{(t_2+1)t_2}$ such quads. The quads of classical type w.r.t. x are determined by y and a point of $\Gamma_1(x) \cap \Gamma_2(y)$, so that there are exactly t_3+1 of these and $\frac{(t_3+1)(t_3-(t_2+1)t_2)}{(t_2+1)t_2}$ of ovoid type.

Given a quad Q of classical type $Q \cap \Gamma_3(x)$ is connected by the lemma in section 0 so that $Q \cap \Gamma_3(x) \subset C$. But $Q \cap \Gamma_3(x)$ contains s^2t_2 points: the point y, $(s-1)(t_2+1)$ neighbours of y and $s^2t_2-1-(s-1(t_2+1)$ nonneighbours of y.

Similarly, given a quad Q of ovoid type, $Q \cap \Gamma_3(x)$ is connected again and contained within C. It contains $s(1+st_2)$ points: the point y, $(s-1)(t_2+1)$ neighbours and $t_2(s^2-s+1)$ nonneighbours. Since any point of $\Gamma_2(y)$ determines a unique quad together with y we proved

$$\begin{aligned} |C \cap \Gamma_2(y)| &\geq (t_3+1)(s^2t_2-1-(s-1)(t_2+1)) + \\ (3) \qquad &+ \frac{(t_3+1)(t_3-t_2(t_2+1))}{t_2+1}(s^2-s+1) \\ &= \frac{(t_3+1)t_3}{t_2+1}(s^2-s+1)-s(t_3+1). \end{aligned}$$

Finally we need a lower bound for $|C \cap \Gamma_3(y)|$. Now if there are e edges going from $\Gamma_2(y)$ to $C \cap \Gamma_3(y)$ then we have $|C \cap \Gamma_3(y)| \geq e/(t_3+1)$. Let us count such edges. From each point in $C \cap \Gamma_2(y)$ there are $(t_3-t_2)(s-1)$ edges to $C \cap \Gamma_3(y)$.

If we choose $z \in \Gamma_2(x) \cap \Gamma_2(y) \cap Q$ where Q is one of the quads considered above, then there is a point $z' \in \Gamma_1(z) \cap \Gamma_1(y) \cap C$. Now consider quads Q' containing the line z'z and not meeting $\Gamma_4(x)$. There are t_3/t_2 such quads, t_2+1 of classical type w.r.t. x and $(t_3-t_2(t_2+1))/t_2$ of ovoid type w.r.t. x. How many edges are there in Q' from z to $C \cap \Gamma_3(y)$? If Q = Q' then none.

Otherwise, if Q' is of classical type w.r.t. x then $(t_2-1)s$ and otherwise t_2s. Thus:

$$e \geq (t_3-t_2)(s-1).\ |C \cap \Gamma_2(y)| +$$

$$+ (t_3+1)\big((s-1)(t_2+1)\big)\big(st_2(t_2-1) + s(t_3-t_2(t_2+1)\big)$$

$$+ \frac{(t_3+1)(t_3-(t_2+1)t_2)}{(t_2+1)t_2}\ \big((s-1)t_2\big)\big(s(t_2+1)(t_2-1) + s(t_3-t_2(t_2+2))\big)$$

where the factors $(s-1)(t_2+1)$ and $(s-1)t_2$ are the number of ways the point z can be chosen in Q.

This yields

$$\begin{aligned}
|C \cap \Gamma_3(y)| &\geq (t_3-t_2)(s-1)\left(\frac{t_3}{t_2+1}(s^2-s+1)-s\right) + \\
&\quad + (s-1)(t_2+1)(st_3-2st_2) \\
&\quad + \frac{t_3-(t_2+1)t_2}{t_2+1}\ (s-1)(st_3-2st_2-s) \\
&= \frac{(s-1)t_3}{t_2+1}\ (t_3(s^2+1) - t_2(s^2+s+1)-s)
\end{aligned} \tag{4}$$

Adding up we find

$$\begin{aligned}
|C| &\geq 1 + (s-1)(t_3+1) + \frac{(t_3+1)t_3}{t_2+1}(s^2-s+1) - s(t_3+1) \\
&\quad + \frac{(s-1)t_3}{t_2+1}(t_3(s^2+1) - t_2(s^2+s+1)-s) \\
&= \frac{s^3t_3(t_3-t_2)}{t_2+1}.
\end{aligned}$$

This proves the theorem. □

<u>REMARK</u>. Now that $|C| = \dfrac{s^3t_3(t_3-t_2)}{t_2+1}$ it follows that all inequalities in the above proof are in fact equalities. This means that no geodesic between two points of C meets $\Gamma_4(x)$.

REMARK. As a side result we find that if H is a regular near hexagon with $s > 1$ and $t_2 > 0$ then for any point $\infty \in H$ we have that $\Gamma_3(\infty)$ is connected. The assumption $t_2 > 0$ can be weakened:

PROPOSITION. *Let* H *be a regular near hexagon, and* $\infty \in H$.
(i) *If* $s = 1$ *then* $\Gamma_3(\infty)$ *is totally disconnected.*
(ii) *If* $s > 1$ *then* $\Gamma_3(\infty)$ *is connected unless* $t_2 = 0$ *and* $s = t = 2$.
There are exactly two nonisomorphic generalized hexagons GH(2,2); *in one of them (the* $G_2(2)$ *hexagon)* $\Gamma_3(\infty)$ *is connected, in the other (its dual) it has two components.*

PROOF. (i) is clear. The information on the generalized hexagons GH(2,2) can be found in [3]. Suppose $\Gamma_3(\infty)$ is disconnected. Then its largest eigenvalue $(s-1)(t_3+1)$ occurs with multiplicity at least two, and by interlacing it follows that $\lambda \geq (s-1)(t_3+1)$ if λ is the second largest eigenvalue of H. Some simple calculations show that this is true only when ($s = 1$ or) $(s,t,t_2) = (2,1,0)$, $(2,2,0)$ or $(2,2,1)$. The first case corresponds to the unique GH(2,1) and the last case to the Hamming cube 3^3. Both have connected $\Gamma_3(\infty)$. □

In a similar way one can show:

PROPOSITION. *Let* O *be a regular near octagon, and* $\infty \in O$.
(i) *If* $s = 1$ *then* $\Gamma_4(\infty)$ *is totally disconnected.*
(ii) *If* $s > 1$ *then* $\Gamma_4(\infty)$ *is connected except if* O *is the unique generalized octagon* GO(2,1) *(now* $\Gamma_4(\infty)$ *has two components) or perhaps if* O *is some generalized octagon* GO(2,4).

2. HEXES

Let (X,L) be a regular near n-gon with $s > 1$ and $t_2 > 0$.

If $x,y \in X$ and $d(x,y) = 3$ then let $H(x,y) := \{u \mid S(x,u) \subset S(x,y)\}$. Clearly $\{x,y\} \subset H(x,y)$. Our aim is to show that $H(x,y)$ is a geodetically closed near hexagon.

(i) If $u \in H(x,y)$ then $d(x,u) \leq 3$.
(For: if $d(x,u) = i$ then $|S(x,u)| = t_i+1$ and $t_i > t_3$ for $i > 3$.)

(ii) $H(x,y) \cap \Gamma_3(x)$ is a component of $\Gamma_3(x)$.
(For: let C be the component of y in $\Gamma_3(x)$. Then $C \subset H(x,y)$ by the first lemma of section 1. But by the proof of the theorem above, given three lines on x, not in a quad, there is a unique subspace $S(x,u)$ with $u \in \Gamma_3(x)$ containing them, and a unique component C such that $S(x,v) = S(x,u)$ for $v \in C$.)

(iii) $H(x,y)$ is a subspace.
(This follows immediately from the definition.)

(iv) $H(x,y)$ contains all geodesics between x and a point $u \in H(x,y)$.
In particular, $H(x,y)$ contains all quads Q and Q' considered in the proof of the theorem above, so that

(v) $H(x,y)$ contains all geodesics between two points $u,v \in H(x,y) \cap \Gamma_3(x)$.

(vi) $H(x,y)$ has diameter 3.

PROOF. Let $u,v \in H(x,y)$. We have to prove $d(u,v) \leq 3$. If $u = x$ or $v = x$ this is true. If $u \in \Gamma_1(x)$ and $d(x,v) \leq 2$ it is also true. If $u \in \Gamma_1(x)$ and $v \in \Gamma_3(x)$ then there is a point at distance 2 from v on the line xu (by definition of $H(x,y)$) and $d(u,v) \leq 3$.

If $u \in \Gamma_2(x)$ and $v \in \Gamma_2(x)$ then choose lines ℓ through u and m through v meeting $H(x,y) \cap \Gamma_3(x) =: C$. If there is a point on ℓ at distance at most two to v then $d(u,v) \leq 3$. But C has diameter 3, and $|\ell \cap C|$, $|m \cap C| \geq 2$ so if no point on ℓ has distance at most two to v then ℓ and m are parallel at distance 2: for any point of ℓ there is a unique point at distance two on m. Contradiction. If $u \in \Gamma_2(x)$ and $v \in \Gamma_3(x)$ then choosing a line ℓ on u meeting C we find $d(u,v) \leq 3$.

Finally, if $u,v \in \Gamma_3(x)$ then $u,v \in C$ and we know already that C has diameter 3. □

(vii) $H(x,y)$ is Cameron closed, i.e., if a point z has two neighbours in $H(x,y)$ then $z \in H(x,y)$.

PROOF. Suppose $z \sim u,v \in H(x,y)$. If $z \in \Gamma_4(x)$ then $u,v \in C$ and $z \in C$ since C is Cameron closed. If $z \in \Gamma_3(x)$ and $u \in C$ then $z \in C$ by definition of component. If $z \in \Gamma_3(x)$ and $u,v \in \Gamma_2(x)$ then either $S(x,u) \neq S(x,v)$ and from $S(x,u) \cup S(x,v) \subset S(x,z)$ it follows that $z \in C$, or $S(x,u) = S(x,v)$, and x,u,v determine a quad. But quads are Cameron closed, contradiction.

If $z \in \Gamma_2(x)\backslash H(x,y)$ then, since $\Gamma_2(x) \cap H(x,y)$ is a union of components of $\Gamma_2(x)$ it follows that $u,v \in \Gamma_1(x)$ and $z \in Q(x,u,v)$, contradiction.

Finally, if $z \in \Gamma_1(x)\backslash H(x,y)$ then $u = v = x$, contradiction. □

(viii) Any point of $H(x,y)$ is on exactly t_3+1 lines within $H(x,y)$.

PROOF. This is clear for the point x, and for points of C.

Let $u \in H(x,y) \cap \Gamma_1(x)$. Let ℓ be a line in $H(x,y)$ on u, $\ell \neq xu$. Then we find a quad $Q(x,u,\ell)$. In this quad both u and x are on t_2 lines different from xu, so the number of lines on u equals the number of lines on x. Similarly for $v \in H(x,y) \cap \Gamma_2(x)$. □

(ix) Let $\gamma_i(u) = |\Gamma_i(u) \cap H(x,y)|$. Then if $u \in H(x,y)$ we have

$$\gamma_0(u) = 1,\ \gamma_1(u) = s(t_3+1),\ \gamma_2(u) = s^2(t_3+1)t_3/(t_2+1),$$

$$\gamma_3(u) = \frac{s^3t_3(t_3-t_2)}{t_2+1}\ .$$

PROOF. The first is obvious, we just proved the second, the third follows from (vii) and (viii) and the last by subtraction (for we know $\gamma_3(x) = |C|$). □

(x) $H(x,y)$ is geodetically closed.

PROOF. Let $u,v \in H(x,y)$, $d(u,v) = 3$. From the value of $\gamma_3(u)$ it follows that v has exactly t_3+1 neighbours in $\Gamma_2(u) \cap H(x,y)$. □

(xi) $H(x,y)$ is a near hexagon.

PROOF. Let u be a point and ℓ a line in $H(x,y)$. Then there is a unique point on ℓ closest to u and distances in $H(x,y)$ are the same as distances in (X,L). □

This completes the proof of our main theorem:

MAIN THEOREM. *Let* (X,L) *be a regular near polygon with* $s > 1$ *and* $t_2 > 0$. *Then any two points at distance three determine a unique geodetically closed sub near hexagon (called a hex).*

COROLLARY. *Under the same hypotheses, any three concurrent lines not in a quad determine a unique hex.*

PROOF. Let ℓ_1,ℓ_2,ℓ_3 be lines through x. Choose $y \in \ell_1\backslash\{x\}$ and $z \in Q(\ell_2,\ell_2)$ with $d(x,z) = 2$. Then $H(y,z)$ is the required hex. $\square$

REMARK. From the existence of hexes one can derive numerous new divisibility conditions on the parameters. Obvious ones are for example

$$(t_3-t_2) \mid (t-t_2),$$
$$t_3(t_3-t_2) \mid t(t-t_2),$$
$$(t_3+1)t_3(t_3-t_2) \mid (t+1)t(t-t_2).$$

For a more detailed discussion see [1].

Thus it follows that near octagons with parameter sets (s,t_2,t_3,t) = (2,1,11,39) or (2,2,14,54) as hypothesized in [4] do not exist.

Another obvious but powerful remark is that a near polygon with parameters $(s,t_2,t_3,\ldots,t)$ can exist only if near hexagons with parameters (s,t_2,t_3) exist. (Of course always assuming $s > 1$, $t_2 > 0$.)

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